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The effect of a shear flow on convection in a layer heated non-uniformly from below

By I. C. WALTON
Department of Mathematics, Imperial College, London SW7†

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In an earlier paper (Walton 1982) we showed that, when a fluid layer is heated non-uniformly from below in such a way that the vertical temperature difference maintained across the layer is a slowly varying monotonic function of a horizontal coordinate x, then convection occurs for $x > x_c$, where x_c is the point where the local Rayleigh number is equal to the critical value for a uniformly heated layer. Furthermore, the amplitude of the convection increases smoothly from exponentially small values for $x \ll x_c$ and asymptotes to a value given by Stuart–Watson theory for $x \gg x_c$.

At the present time no solutions of this kind are available for a class of problems in which the onset of instability is heavily influenced by a shear flow (e.g. Görtler vortices in a boundary layer on a curved wall, convection in a heated Blasius boundary layer). In a first step to bridge the gap between these problems and in order to elucidate the difficulties associated with the presence of a shear flow, we investigate the effect of a (weak) shear flow on our earlier convection problem. We show that the onset of convection is delayed and that it appears more suddenly, but still smoothly. The role of horizontal diffusion is shown to be of paramount importance in enabling a solution of this kind to be found, and the implications of these results for instabilities in higher-speed shear flows are discussed.

1. Introduction

The stability of fluid flow is determined by considering the effect of an arbitrary infinitesimally small disturbance on the flow. If all such disturbances decay with time the flow is said to be stable, while if any disturbance grows with time the flow is said to be unstable. Often there exists in the problem a dimensionless parameter, denoted here by Ra, which characterizes the flow and provides a guide to its stability. Typically, if Ra is below a critical value, Ra_c say, the flow is stable, while if $Ra > Ra_c$ it is unstable. The value of Ra_c is determined on the basis of linearized perturbation theory in which a time dependence of the form $e^{\sigma t}$ is assumed. If $\mathrm{Re}\{\sigma\} < 0$ the flow is stable, and if $\mathrm{Re}\{\sigma\} > 0$ it is unstable. The point $Ra = Ra_c$ where $\mathrm{Re}\{\sigma\} = 0$ or $\mathrm{d}A/\mathrm{d}t = 0$, where A is the amplitude of a typical perturbed flow component, is known as the point of neutral stability or the critical point.

It is often the case (for example in cellular convection or circular Couette flow) that for values of Ra close to Ra_c the amplitude A is governed by an equation of the form

$$\frac{\mathrm{d}A}{\mathrm{d}t} = (Ra - Ra_c)\,A - A^3, \tag{1.1}$$

† Present address: BP Research Centre, Chertsey Road, Sunbury on Thames, Middlesex.

derived on the basis of weakly nonlinear theory (see e.g. Stuart 1971; DiPrima 1978). Here the amplitude A is assumed to be finite but small. Solutions of (1.1) take the form

$$A^2 = \frac{C(Ra - Ra_c)\exp\{2(Ra - Ra_c)t\}}{1 + C\exp\{2(Ra - Ra_c)t\}}, \tag{1.2}$$

where C is a constant of integration. If the disturbance is generated by an infinitesimally small perturbation of the base flow at time $t = -\infty$ then, since A is now finite, we require $A \to 0$ as $t \to -\infty$. For $Ra < Ra_c$ the only possible solution is $C = 0$, in which case the amplitude remains zero for all time. On the other hand, if $Ra > Ra_c$ all solutions satisfy the initial condition, and all approach $(Ra - Ra_c)^{\frac{1}{2}}$ as $t \to +\infty$ (see figure 1).

Alternatively one can consider the fate of a small but finite-amplitude disturbance imposed at some finite point in time, say $t = 0$. If A_0 denotes the initial amplitude then (1.2) gives

$$C = \frac{A_0^2}{Ra - Ra_c - A_0^2}.$$

If $Ra < Ra_c$ the amplitude decays to zero as $t \to \infty$, while it approaches the finite value $(Ra - Ra_c)^{\frac{1}{2}}$ as $t \to \infty$ if $Ra > Ra_c$ (see figure 2). These two points of view point to the same end: Ra_c is the critical value of Ra that separates a regime in which only the base flow is present from one in which a weak finite-amplitude secondary flow is also present.

The foregoing considerations apply to a base flow that depends upon only one spatial variable, say z, and are very familiar. But consider now a base flow that varies with z and with one other spatial variable, say x, and suppose that the variation with x is sufficiently weak that to leading order the linearized perturbation equations contain no x-derivatives but merely contain x as a parameter. Then the stability at any point depends only upon local properties of the flow, and we may divide the flow regime into a stable part (say $x < x_c$) and unstable part ($x > x_c$) using either criterion for (linear) stability discussed above. The location $x = x_c$ may be described as a point of neutral stability. A complication arises when the basic flow is non-parallel, because, as several authors (e.g. Bouthier 1972; Eagles & Weissman 1975) have pointed out, consideration of x-derivatives (even at higher order) leads to the result that different flow quantities have different points of neutral stability.

There are two approaches towards the variation of the amplitude of the disturbance with x in the neighbourhood of $x = x_c$ in the steady state. The first has been adopted by Hall & Smith (1984), who considered the growth of Tollmien–Schlichting waves in a Blasius boundary layer, and by Hall (1982), who considered the development of Görtler vortices in a non-parallel boundary layer on a curved wall. These authors obtained amplitude equations that took the general form

$$\frac{dA}{dX} = XA - A^3, \tag{1.3}$$

where X is proportional to $x - x_c$. This equation is similar to (1.1) in that the term in A^3 arises from nonlinear interactions, and the term XA, due to the variation in a local stability parameter about critical, plays the same role as does $(Ra - Ra_c)A$ in (1.1). The X-derivative arises through the inertial terms and appears because of a base velocity in the x-direction. It is a straightforward matter to demonstrate that all solutions of (1.3) become singular at some finite value of X.† A typical solution

† The general solution is $A = e^{\frac{1}{2}X^2}/[2\int_{X_0}^{X} e^{t^2}\,dt]^{\frac{1}{2}}$, where X_0 is an arbitrary constant.

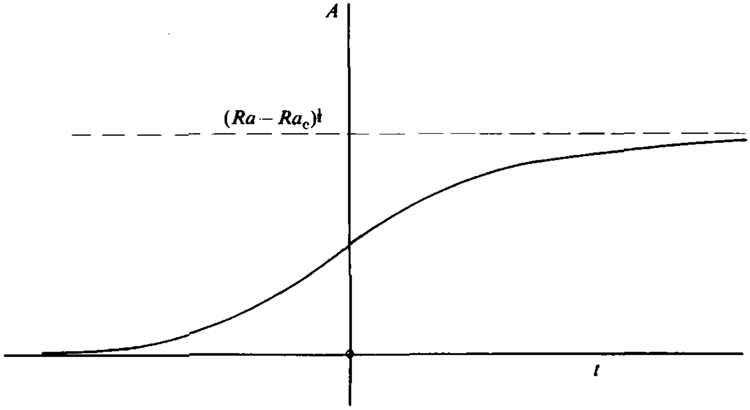

FIGURE 1. The amplitude $|A|$ given by (1.2) for $Ra > Ra_c$.

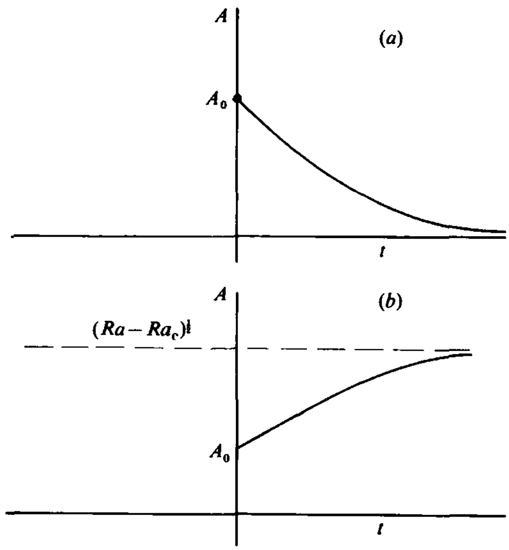

FIGURE 2. The amplitude $|A|$ given by (1.2) with $|A| = |A_0|$ at $t = 0$ for (a) $Ra < Ra_c$, (b) $Ra > Ra_c$.

curve, which asymptotes to $X^{\frac{1}{2}}$ as $X \to +\infty$, is shown in figure 3. If the point of neutral stability is defined, as in the time-dependent problem, to be that point where $\mathrm{d}A/\mathrm{d}X = 0$, then, since this occurs where the amplitude curve intersects the parabola $A^2 = X$, this point is not coincident with that given by linear theory.

A defect of this solution is that it cannot satisfy the boundary condition that needs to be imposed on A if the disturbance is generated by either of the mechanisms described above. For suppose that the flow is given either an infinitesimally small disturbance maintained for all time or else a small but finite-amplitude disturbance at some time $t = 0$. In both cases the finite-amplitude steady state is zero in the stable region far away from the point of neutral stability $X = 0$. Therefore we require $A \to 0$ as $X \to -\infty$, but there is no solution of (1.3) with this property. What this solution can describe, though, is the response of the fluid to a finite-amplitude disturbance, say $A = A^*$ at $X = X^*$, when this disturbance is maintained for all time. This is the

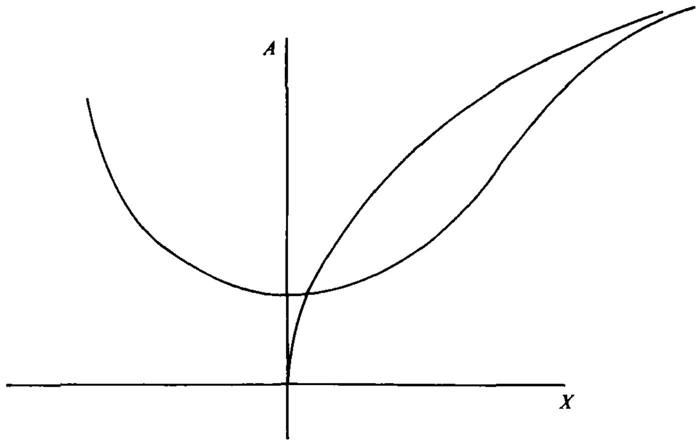

FIGURE 3. The amplitude A given by (1.3).

steady state that one would expect to find downstream of a generator in an experiment. While certain flow instabilities are generated in this way and there is no doubt that this is a concern of some importance, many other flows develop instabilities 'naturally' or 'spontaneously', and it is clear that such instabilities cannot be discussed within this framework. Two further points are worth making about this solution. First, not only does the point of neutral stability depend upon the flow quantity under consideration, but it also depends upon the magnitude and location of the generating disturbance. Second, (1.3) is obtained by achieving a balance of the three terms by suitable choice of scales of X, and A in terms of ϵ ($\ll 1$), a measure of the relative weakness of the x-variations, and a Reynolds number Re (usually $\gg 1$) (ϵ, Re may not be independent in some circumstances). Consequently only forced disturbances of a certain magnitude, depending in a precise way on ϵ, Re, can be accommodated by this theory, and the formula is severely restricted. We shall return to this point in §4.

An alternative approach towards the finite-amplitude solution in the neighbourhood of the point of neutral stability has been taken by Walton (1982, hereinafter referred to as I), who considered the stability of a horizontal layer of fluid of depth h heated non-uniformly from below in such a way that the temperature difference between the boundaries increased monotonically with a horizontal variable x. Provided that the horizontal scale L of the variation in the boundary temperature is much larger than h (i.e. $\epsilon = h/L \ll 1$), the leading-order approximations to the perturbation quantities depend only upon x as a parameter. Then according to linearized theory the point of neutral stability, x_c, is defined to be that value of x at which a local Rayleigh number Ra_x is equal to Ra_c, the critical value of Ra for a uniformly heated layer. In the neighbourhood of $x = x_c$ a steady-state nonlinear amplitude equation similar to (1.3) may be derived, except that, since there is no base velocity in the x-direction to the order considered, there is no term in $\mathrm{d}A/\mathrm{d}x$. Instead, the right-hand side is balanced by the leading x-derivatives, which stem from the diffusion terms. For longitudinal rolls (aligned with their axes in the x-direction), the amplitude A satisfies

$$\frac{\mathrm{d}^4A}{\mathrm{d}X^4} = XA - A^3, \tag{1.4}$$

where $X \propto x - x_c$. There is no difficulty in obtaining a solution of (1.4) that satisfies boundary conditions consistent with an infinitesimal disturbance of the base flow or

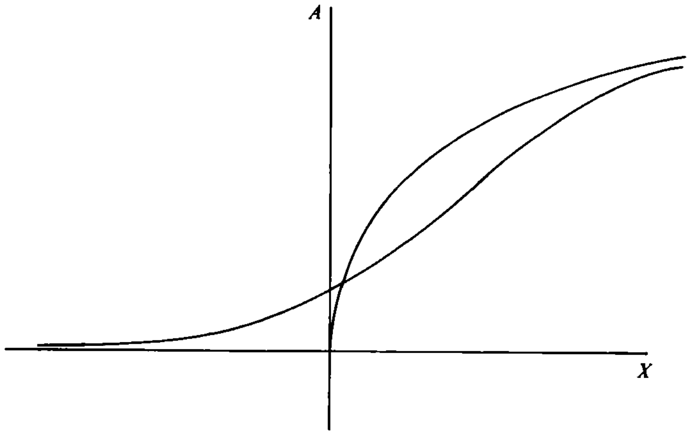

FIGURE 4. The amplitude A given by (1.4).

a small initial perturbation. The appropriate solution, shown in figure 4, decays to zero as $X \to -\infty$ and approaches the parabola $A^2 = X$ as $X \to +\infty$. There is no finite value of X at which $\mathrm{d}A/\mathrm{d}X = 0$ (except in the exponentially decaying 'tail'), and the concept of a point of neutral stability is destroyed.

The flows discussed by Hall & Smith (1984) and Hall (1982) are quite different from that discussed by Walton (1982) in that the former are heavily influenced by a shear flow in the x-direction, while none is present in the latter. Moreover, no attempt is made in the former studies to take account of diffusion in the x-direction, but it plays a major role in the latter work. The questions naturally arise of the effect of a shear flow on the configuration discussed by Walton and of the effect of diffusion in Hall & Smith's and Hall's studies. In particular, is it possible to find an amplitude equation that possesses a solution that decays to zero as $X \to -\infty$ even when a shear flow is present? As a first step in answering these questions we shall consider the onset of thermal convection in a non-uniformly heated layer that is subject to a weak shear flow in the x-direction. We restrict our attention to shear-flow Reynolds numbers that are in some sense (to be defined more precisely later) small. While we acknowledge that the effect of large-Reynolds-number flows may be different, the analysis presented here does, it is hoped, point us in the right direction.

The equations that govern this flow are set up in §2, which also contains a discussion of the steady base flow set up by the horizontal variations in the temperature distribution. Perturbations to this state are discussed in §3, leading to the derivation and solution of the amplitude equations. The results are discussed in §4.

2. Base flow

An incompressible Boussinesq fluid is contained between horizontal boundaries $z^* = 0, h$ maintained at temperatures $T_0^* + \Delta T(1 + F(\epsilon x^*/h))$ and T_0^* respectively, where $\Delta T > 0$, x^* is a horizontal coordinate, ϵ a small parameter and F a prescribed function of $\epsilon x^*/h$, restricted here to have positive gradient at $x^* = 0$. In addition, a shear flow is generated either by an imposed pressure gradient in the x^* direction (Poiseuille flow) or by allowing the horizontal boundaries to move with suitable velocities in the x^* direction (Couette flow).

We take ΔT as a typical temperature scale, h as a lengthscale, $\Delta T\,\alpha g h\rho$ as a pressure scale and h^2/k as a timescale, where α is the coefficient of volume expansion, g is the acceleration due to gravity and k is the coefficient of thermal diffusivity. The instability that forms the subject of this paper is thermal in origin, and it is convenient to use a velocity scale derived from buoyancy effects (as in Rayleigh–Bénard convection) in scaling the departure of the velocity field from the imposed shear flow. The dimensional velocity field $\boldsymbol{u}^*$ is written as

$$\boldsymbol{u}^* = U_0\,U(z)\,\hat{\boldsymbol{x}}^* + \frac{\alpha g\,\Delta T\,h^2}{\nu}\boldsymbol{u},$$

where ν is the kinematic viscosity, U_0 is a velocity scale typical of the imposed shear flow and $U(z)$ and $\boldsymbol{u}$ denote the dimensionless imposed shear flow and augmented velocity field respectively. Also,

$$T^* = T_{\text{ref}} + T\,\Delta T \quad \text{and} \quad p^* = (\nu U_0\rho/h)\,P + \Delta T\,\alpha\rho g h p, \quad \text{where} \quad \partial P/\partial x = \mathrm{d}^2U/\mathrm{d}z^2.$$

In dimensionless form the equations governing the flow are

$$\left.\begin{aligned} Pr^{-1}\frac{\partial \boldsymbol{u}}{\partial t} + Pr^{-1}\,Ra(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u} + Re\left(U\frac{\partial \boldsymbol{u}}{\partial x} + w\frac{\mathrm{d}U}{\mathrm{d}z}\,\hat{\boldsymbol{x}}\right) &= -\boldsymbol{\nabla}p + T\hat{\boldsymbol{z}} + \nabla^2\boldsymbol{u}, \\ \frac{\partial T}{\partial t} + Ra\,(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,T + Pr\,Re\,U\frac{\partial T}{\partial x} &= \nabla^2 T, \\ \boldsymbol{\nabla}\cdot\boldsymbol{u} &= 0, \end{aligned}\right\} \quad (2.1)$$

where T, p and t are the dimensionless temperature, pressure and time respectively. The dimensionless numbers that appear in these equations are defined by

$$\begin{aligned} Pr &= \nu/k \quad \text{(Prandtl number)}, \\ Ra &= \alpha g\,\Delta T\,h^3/\nu k \quad \text{(Rayleigh number)}, \\ Re &= U_0 h/\nu \quad \text{(Reynolds number)}. \end{aligned}$$

The boundary conditions on T and $\boldsymbol{u}$ are those for constant-temperature no-slip boundaries at rest. Assuming a temperature variation on the lower boundary of the form described above, we have

$$\left.\begin{aligned} \boldsymbol{u} = 0 \quad \text{at } z = 0, 1, \\ T = 0 \quad \text{at } z = 1, \quad T = 1 + F(\epsilon x) \quad \text{at } z = 0. \end{aligned}\right\} \quad (2.2)$$

In classical Rayleigh–Bénard convection in which the lower boundary is heated uniformly ($F \equiv 0$) the state that exists prior to the onset of cellular convection is motionless and heat is transferred to the upper boundary by conduction alone. However, horizontal variations in the thermal boundary conditions (2.2) induce a steady circulation which needs to be determined before we investigate the onset of cellular convection.

We assume that the base state is steady ($\partial/\partial t \equiv 0$), two-dimensional ($\partial/\partial y \equiv 0$) and takes place on the horizontal lengthscale typical of the variation in the temperature of the lower boundary. Let us write

$$(\boldsymbol{u}, p, T) = (\boldsymbol{u}_{\mathrm{B}}, p_{\mathrm{B}}, T_{\mathrm{B}})\,(X, z),$$

where $\boldsymbol{u}_{\mathrm{B}} = u_{\mathrm{B}}\,\hat{\boldsymbol{x}} + w_{\mathrm{B}}\,\hat{\boldsymbol{z}}$ and $X = \epsilon x$.

Then (2.1) gives

$$\left.\begin{aligned} \frac{Ra}{Pr}\left[\epsilon u_{\mathrm{B}}\frac{\partial u_{\mathrm{B}}}{\partial X}+w_{\mathrm{B}}\frac{\partial u_{\mathrm{B}}}{\partial z}\right]+Re\left[\epsilon U\frac{\partial u_{\mathrm{B}}}{\partial X}+w_{\mathrm{B}}\frac{\mathrm{d}U}{\mathrm{d}z}\right] &= -\epsilon\frac{\partial p_{\mathrm{B}}}{\partial X}+\nabla^2 u_{\mathrm{B}}, \\ \frac{Ra}{Pr}\left[\epsilon u_{\mathrm{B}}\frac{\partial w_{\mathrm{B}}}{\partial X}+w_{\mathrm{B}}\frac{\partial w_{\mathrm{B}}}{\partial z}\right]+Re\left[\epsilon U\frac{\partial w_{\mathrm{B}}}{\partial X}\right] &= -\frac{\partial p_{\mathrm{B}}}{\partial z}+T_{\mathrm{B}}+\nabla^2 w_{\mathrm{B}}, \\ Ra\left[\epsilon u_{\mathrm{B}}\frac{\partial T_{\mathrm{B}}}{\partial X}+w_{\mathrm{B}}\frac{\partial T_{\mathrm{B}}}{\partial z}\right]+Re\left[\epsilon U\frac{\partial T_{\mathrm{B}}}{\partial X}\right] &= \nabla^2 T_{\mathrm{B}}, \\ \epsilon\frac{\partial u_{\mathrm{B}}}{\partial X}+\frac{\partial w_{\mathrm{B}}}{\partial z} &= 0, \end{aligned}\right\} \tag{2.3}$$

where $$\nabla^2 \equiv \frac{\partial^2}{\partial z^2}+\epsilon^2\frac{\partial^2}{\partial X^2}.$$

The horizontal scale of the non-uniform heating is assumed to be large compared with the height of the fluid layer, which means $\epsilon \ll 1$. Also in this paper we shall assume that the imposed shear flow is weak, so that $Re \ll 1$. Under these conditions a solution of (2.3) and (2.2) may be sought by expanding in powers of ϵ and Re. Let us write

$$u_{\mathrm{B}} = \epsilon u_0+O(\epsilon^3, \epsilon^2 Re), \quad w_{\mathrm{B}} = \epsilon^2 w_0+O(\epsilon^4, \epsilon^3 Re),$$
$$T_{\mathrm{B}} = T_0+O(\epsilon^2, \epsilon Re), \quad p_{\mathrm{B}} = p_0+O(\epsilon^2, \epsilon Re).$$

Then we find that $$\frac{\partial^2 T_0}{\partial z^2} = 0,$$

which together with (2.2) gives

$$T_0 = (1-z)(1+F(X)). \tag{2.4}$$

Also $$\frac{\partial p_0}{\partial z} = T_0, \quad \frac{\partial p_0}{\partial X} = \frac{\partial^2 u_0}{\partial z^2}, \quad \frac{\partial w_0}{\partial z} = -\frac{\partial u_0}{\partial X},$$

which, after some algebra, yields

$$w_0 = w_{01}(z)\,F''(X), \quad u_0 = -w_{01}'(z)\,F'(X),$$

where $w_{01}(z) = \frac{1}{120}(2z^5-5z^4+4z^3-z^2)$.

To leading order the interaction between the shear flow and the buoyancy-driven circulation is unimportant. It remains unimportant as long as $Re \ll \epsilon^{-1}$, so the base state determined here is of much wider validity than is needed for our present purposes.

3. Perturbations of the base state

Rayleigh–Bénard convection occurs in a uniformly heated layer when the Rayleigh number Ra exceeds a critical value Ra_{c} and the horizontal wavenumber k is equal to k_{c}. For non-slip boundaries it is known that $Ra_{\mathrm{c}} = 1707.762$ and $k_{\mathrm{c}} = 3.1172$.

When the temperature difference between the boundaries increases slowly but monotonically with horizontal distance, as it does here, we may introduce the concept of a 'local' Rayleigh number Ra_x defined in terms of the local vertical temperature difference by

$$Ra_x = \frac{\alpha g \Delta T (1+F(X))\, h^3}{\nu k} = Ra\,(1+F(X)).$$

It is convenient to locate the origin of the coordinate system at that point on the lower boundary where the local Rayleigh number is exactly equal to the critical value, which means that

$$Ra_x = Ra_c\,(1+F(X)).$$

Then, if $F(0) = 0$ and F is monotonically increasing with X, $Ra_x \gtrless Ra_c$ for $X \gtrless 0$. If derivatives with respect to X are ignored we may deduce on the basis of linearized perturbation theory that the base state is stable for $X < 0$ and unstable for $X > 0$. Linearized theory is not, however, adequate to describe the solution in the unstable region, for the disturbance grows rapidly and nonlinear effects need to be taken into account. The solution in the neighbourhood of $X = 0$ was given in I for $Re = 0$. The most important feature of the analysis is the adoption of a new horizontal variable, intermediate between the 'fast' (x) and 'slow' (X) variables, over which the effects of weak nonlinearity are balanced by diffusive terms (in the x-direction) and by terms due to the departure of the local Rayleigh number from critical. The amplitude equations that result have solutions that increase smoothly from exponentially small values towards the stable end of a narrow 'transition' region near $X = 0$ and take on the square-root behaviour typical of weakly nonlinear theory at the stable end. The object of the present calculation is to extend these results to $Re \neq 0$ but small.

As in I we shall focus our attention on perturbations in the form of rolls aligned with their axes in the y-direction (transverse rolls) or in the x-direction (longitudinal rolls). Let us write

$$\boldsymbol{u} = \boldsymbol{u}_B + \bar{\boldsymbol{u}}, \quad T = T_B + \bar{\theta}, \quad p = p_B + \bar{p},$$

where $\bar{\boldsymbol{u}}$, $\bar{\theta}$ and $\bar{p}$ are the perturbed variables. Then $\bar{u}$, $\bar{\theta}$ and $\bar{p}$ satisfy

$$\left.\begin{aligned} Pr^{-1}\frac{\partial \bar{\boldsymbol{u}}}{\partial t} + \frac{Ra_c}{Pr}[(\boldsymbol{u}_B\cdot\boldsymbol{\nabla})\,\bar{\boldsymbol{u}} + (\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla})\,\boldsymbol{u}_B + (\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla})\,\bar{\boldsymbol{u}}] + Re\left[U\frac{\partial \bar{\boldsymbol{u}}}{\partial x} + \bar{w}\frac{\mathrm{d}U}{\mathrm{d}z}\hat{\boldsymbol{x}}\right] \\ = -\bar{\boldsymbol{\nabla}}\bar{p} + \bar{\theta}\hat{\boldsymbol{z}} + \nabla^2\bar{\boldsymbol{u}}, \\ \frac{\partial \bar{\theta}}{\partial t} + Ra_c\,[(\boldsymbol{u}_B\cdot\boldsymbol{\nabla})\,\bar{\theta} + (\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla})\,T_B] + Re\,Pr\,U\frac{\partial \bar{\theta}}{\partial x} = \nabla^2\bar{\theta}, \\ \boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}} = 0, \end{aligned}\right\} \quad (3.1)$$

and

$$\bar{\boldsymbol{u}} = \bar{\theta} = 0 \quad \text{at } z = 0, 1.$$

3.1. *Transverse rolls*

It was shown in I that for $Re = 0$ the amplitude of the transverse mode is $O(\epsilon^{\frac{1}{3}})$ and varies in the x-direction on a lengthscale $O(h/\epsilon^{\frac{1}{3}})$, so that it depends on the intermediate variable $X_T = \epsilon^{-\frac{2}{3}}X = \epsilon^{\frac{1}{3}}x$. Furthermore, the solution is expanded in powers of $\epsilon^{\frac{1}{3}}$, and the amplitude equation arises by considering terms $O(\epsilon)$.

For classical Rayleigh–Bénard convection in the presence of a weak Couette flow, Ingersoll (1966) has shown that the critical Rayleigh number for transverse rolls is greater by a term in Re^2 than that for $Re = 0$.†

These results suggest that the effect of the shear flow is of the same order of magnitude as the other effects that contribute to the amplitude equation when $\epsilon^{\frac{1}{3}} Re^2 \sim \epsilon$,

† A similar result holds for Poiseuille flow (Walton 1985).

i.e. $Re \sim \epsilon^{\frac{1}{3}}$. Let us write $Re = Re_T\,\epsilon^{\frac{1}{3}}$ and suppose for now that $Re_T \sim 1$. We expand the perturbation variables in powers of $\epsilon^{\frac{1}{3}}$ as follows:

$$\begin{pmatrix} \bar{u} \\ \bar{\theta} \\ \bar{p} \end{pmatrix} = \epsilon^{\frac{1}{3}} E A_T(X_T) \begin{pmatrix} \bar{u}_1 \\ \bar{\theta}_1 \\ \bar{p}_1 \end{pmatrix} + \text{c.c.} + \epsilon^{\frac{2}{3}} \left[E^2 A_T^2 \begin{pmatrix} \bar{u}_{21} \\ \bar{\theta}_{21} \\ \bar{p}_{21} \end{pmatrix} + |A_T|^2 \begin{pmatrix} \bar{u}_{22} \\ \theta_{22} \\ p_{22} \end{pmatrix} \right.$$

$$\left. + 2\mathrm{i}k_c \frac{\mathrm{d}A_T}{\mathrm{d}X_T} E \begin{pmatrix} \bar{u}_{23} \\ \bar{\theta}_{23} \\ \bar{p}_{23} \end{pmatrix} + \mathrm{i}Re_T\, k_C\, A_T \begin{pmatrix} \bar{u}_{24} \\ \bar{\theta}_{24} \\ \bar{p}_{24} \end{pmatrix} + \text{c.c.} \right] + \epsilon \left[E \bar{A}_T \begin{pmatrix} \bar{u}_{31} \\ \bar{\theta}_{31} \\ \bar{p}_{31} \end{pmatrix} + \dots \right], \quad (3.2)$$

where $E = \exp\{\mathrm{i}k_c x - \mathrm{i}k_c Pr\, Re_T\, \epsilon^{\frac{1}{3}} ct\}$ and $\bar{v} \equiv 0$. The wave speed c needs to be expanded in powers of ϵ also, i.e.

$$c = c_0 + \epsilon^{\frac{1}{3}} c_1 + \dots,$$

but only c_0 enters our calculations.

When (3.2) is substituted into the perturbation equations (3.1) and terms in $\epsilon^{\frac{1}{3}}$ and $\epsilon^{\frac{2}{3}}$ are equated to zero, we may determine the terms in $\epsilon^{\frac{1}{3}}$ and $\epsilon^{\frac{2}{3}}$ in (3.2), and c_0, while terms in ϵE yield the amplitude equation. Details of the calculation are very similar to those in I; they are not given here but are available on request from the author. The amplitude equation is

$$\alpha_1 A_T |A_T|^2 + \alpha_2 \frac{\mathrm{d}^2 A_T}{\mathrm{d}X_T^2} + \alpha_3 X_T A_T + \alpha_4 Re_T^2 A_T + \alpha_5 Re_T \frac{\mathrm{d}A_T}{\mathrm{d}X_T} = 0, \quad (3.3)$$

where $\alpha_1 < 0$, $\alpha_2 > 0$, $\alpha_3 > 0$, $\alpha_4 < 0$ and $\alpha_5 < 0$. Values of α_i, $i = 1, \dots, 5$, are given in Appendix A. We require A_T to decay to zero as $X_T \to -\infty$, while for $X_T \gg 1$ the nonlinear term is balanced by the term in X_T due to the increment of the local Rayleigh number above critical, which yields

$$A_T \to 0 \quad \text{as } X_T \to -\infty, \quad |A_T| \sim \left(-\frac{\alpha_3 X_T}{\alpha_1}\right)^{\frac{1}{2}} \quad \text{as } X_T \to +\infty. \quad (3.4)$$

Since the coefficients in (3.3) and (3.4) are real we may assume that A_T is real also.

Four of the coefficients in (3.3) may be removed by a canonical transformation similar to that used in I. Let

$$\left.\begin{aligned} A_T &= \left(\frac{\alpha_2}{\alpha_3}\right)^{-\frac{1}{3}} \left(-\frac{\alpha_1}{2\alpha_2}\right)^{-\frac{1}{2}} \bar{A}_T, \\ X_T &= \left(\frac{\alpha_2}{\alpha_3}\right)^{\frac{1}{3}} \bar{X}_T - \left(\frac{\alpha_4}{\alpha_3}\right) Re_T^2. \end{aligned}\right\} \quad (3.5)$$

Then we have

$$\left.\begin{aligned} &\frac{\mathrm{d}^2 \bar{A}_T}{\mathrm{d}\bar{X}_T^2} - \delta \frac{\mathrm{d}\bar{A}_T}{\mathrm{d}\bar{X}_T} + \bar{X}_T \bar{A}_T - 2\bar{A}_T^3 = 0, \\ \text{with} \quad &\bar{A}_T \to 0 \quad \text{as } \bar{X}_T \to -\infty, \quad \bar{A}_T \sim (\tfrac{1}{2}\bar{X}_T)^{\frac{1}{2}} \quad \text{as } \bar{X}_T \to +\infty. \end{aligned}\right\} \quad (3.6)$$

Here $\delta = -\alpha_5 \alpha_2^{-\frac{1}{3}} \alpha_3^{-\frac{2}{3}} Re_T$, which is positive since $\alpha_5 < 0$.

The numerical solution of (3.6) has been given in I for $\delta = 0$, in which case (3.6) is known as the second Painlevé transcendent. This solution holds for $Re_T = 0$ (which is the problem considered in I) and therefore to leading order for $Re_T \ll 1$ or $Re \ll \epsilon^{\frac{1}{3}} h$. It also holds for $Re_T \sim 1$ in the special case of antisymmetric Couette flow $U = z - \frac{1}{2}$, for which $\alpha_5 = 0$. The amplitude then is exactly the same as when no other flow is

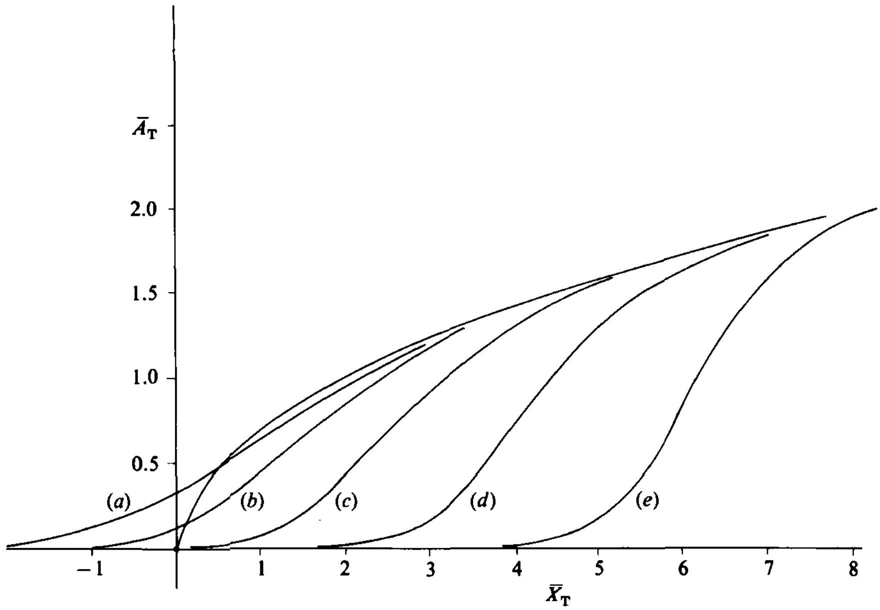

FIGURE 5. The amplitude $\bar{A}_T$ given by (3.6) for values of δ as follows: (a) 0; (b) 1; (c) 2; (d) 3; (e) 4.

imposed, and the only effect of the shear flow is to shift the amplitude curve a distance $(\alpha_4/\alpha_3)\,Re_T^{\frac{2}{3}}$ downstream on the scale of $\bar{X}_T$ (or $(\alpha_4/\alpha_3)\,Re^2\,\epsilon^{-\frac{1}{3}}h$). This result is to be expected once it is known that a shear flow increases the critical Rayleigh number from that for uniformly heated Rayleigh–Bénard convection by a factor proportional to Re^2 for small Re; in this context the onset of convection is delayed until the local Rayleigh number has been increased by $O(Re^2)$, and this requires $X \sim Re^2$ or $\bar{X}_T \sim Re_T^{\frac{2}{3}}$.

Equation (3.6) has been solved numerically for a range of values of δ using the methods described in I, and sample results are shown in figure 5. The solution curves have a similar shape for all values of δ, but as δ increases it appears that the transition region, where the amplitude rises from very small values to close to the parabola $\bar{A}_T = (\frac{1}{2}\bar{X}_T)^{\frac{1}{2}}$, becomes narrower and is shifted to the right. This description may be made more precise as follows. Let us take the value $\bar{X}_{T0}$ of $\bar{X}_T$, where $d^2\bar{A}_T/d\bar{X}_T^2 = 0$, to be the 'centre' of the transition region. For $\delta \gg 1$ we may find an asymptotic form for $\bar{A}_T$ by writing

$$\bar{X}_T = \bar{X}_{T0}\,\delta^2 + \bar{X}_{T1}\,\delta^{-1}, \quad \bar{A}_T = \delta \bar{A}_{T0}.$$

Then to leading order in δ we have

$$\frac{d^2\bar{A}_{T0}}{d\bar{X}_{T1}^2} + \bar{X}_{T0}\,\bar{A}_{T0} - 2\bar{A}_{T0}^3 - \frac{d\bar{A}_{T0}}{d\bar{X}_{T1}} = 0, \tag{3.7}$$

with $\bar{A}_{T0} \to 0$ as $\bar{X}_{T1} \to -\infty$, and $\bar{A}_{T0} \to (\frac{1}{2}\bar{X}_{T0})^{\frac{1}{2}}$ as $\bar{X}_{T1} \to \infty$. The scaling adopted here is the one that balances all four terms in (3.6) as $\delta \to \infty$.

It seems to be possible to obtain solutions of (3.7) for all values of $\bar{X}_{T0}$, one of which ($\bar{X}_{T0} = \frac{2}{9}$) gives the analytic solution

$$\bar{A}_{T0} = \tfrac{1}{2}(1 + \tanh \tfrac{1}{6}\bar{X}_{T1})\,(\tfrac{1}{2}\bar{X}_{T0})^{\frac{1}{2}}. \tag{3.8}$$

The appropriate value of $\bar{X}_{T0}$ may be determined, at least in principle, by specifying more exactly the asymptotic behaviour of $\bar{A}_{T0}$ as $\bar{X}_{T1} \to \pm\infty$ by matching with the solution outside the transition region. As we show in Appendix B, such a condition may only be obtained for $\bar{X}_{T0} < \frac{1}{4}$, but we have been unable to find a solution of (3.7) and the more exact boundary condition in that range. We conclude that $\bar{X}_{T0} > \frac{1}{4}$. Numerical integration of (3.6) for large values of δ indicates that $\bar{X}_{T0} \approx 0.30$, which is at least consistent with our conclusion based on the asymptotic analysis.

These results indicate that the horizontal extent of the transition region diminishes like δ^{-1} as $\delta \to \infty$, which means that its thickness is $O(\epsilon^{\frac{1}{3}}/Re)$ on the scale of $\bar{X}_T$, equivalent to Re^{-1} on the scale of x. Also, the transition becomes localized near where $X_T \sim \delta^2 = Re^2/\epsilon^{\frac{2}{3}}$, which is equivalent to $X \sim Re^{\frac{2}{3}}$ or $x \sim Re^{\frac{2}{3}}/\epsilon$. Thus as Re increases, the onset of convection is delayed (occurs further downstream) and is more sudden. When Re is as large as unity, it appears that the onset of convection occurs at an $O(1)$ distance downstream (on the scale of X) and that transition takes place on a lengthscale $O(1)$ (on the scale of x), or within a few wavelengths. Under these circumstances the intermediate variable $\bar{X}_T$ is indistinguishable from the fast variable x, multiple-scaling techniques are no longer applicable, and it is necessary to treat the full partial differential equations. This is outside the scope of the present paper.

3.2. *Longitudinal rolls*

It was shown in I that for $Re = 0$ the amplitude of the longitudinal mode is $O(\epsilon^{\frac{2}{5}})$ and varies in the x-direction on a lengthscale $O(h/\epsilon^{\frac{2}{5}})$, so that it depends upon the intermediate variable $X_L = \epsilon^{-\frac{4}{5}}X = \epsilon^{+\frac{1}{5}}x$. The solution is expanded in powers of $\epsilon^{\frac{2}{5}}$, and the amplitude equation arises by considering terms $O(\epsilon^{\frac{6}{5}})$.

It is known that the stability of a uniformly heated layer to longitudinal rolls is unaffected by the shear flow, but this is not the case for non-uniform heating. Here the leading effect of the shear flow is manifest through advective terms, which are proportional to $\epsilon\, Re\, U\, \mathrm{d}/\mathrm{d}X$ or $Re\, U \epsilon^{\frac{1}{5}}\, \mathrm{d}/\mathrm{d}X_L$. For transverse rolls the leading effect (again due to advection) appeared through terms in $\mathrm{i}\, Re\, U k_c$, which were balanced by terms involving the phase speed of the disturbance. Here the leading contribution is a real quantity, which cannot be balanced by a phase speed. We shall consider its effect on the amplitude equation by writing $Re = \epsilon^{\frac{3}{5}}\, Re_L$ with $Re_L \sim 1$.

As in I, we expand the solution in powers of $\epsilon^{\frac{2}{5}}$;

$$\begin{pmatrix} \bar{v} \\ \bar{w} \\ \bar{\theta} \end{pmatrix} = \epsilon^{\frac{2}{5}} A_L(X_L)\, \mathrm{E} \begin{pmatrix} \bar{v}_1 \\ \bar{w}_1 \\ \bar{\theta}_1 \end{pmatrix} + \text{c.c.} + \epsilon^{\frac{4}{5}} |A_L|^2 \begin{pmatrix} \bar{v}_{21} \\ \bar{w}_{21} \\ \bar{\theta}_{21} \end{pmatrix} + \epsilon^{\frac{4}{5}} A_L^2 E^2 \begin{pmatrix} \bar{v}_{22} \\ \bar{w}_{22} \\ \bar{\theta}_{22} \end{pmatrix}$$

$$+ \epsilon^{\frac{4}{5}} \frac{\mathrm{d}^2 A_L}{\mathrm{d}X_L^2} E \begin{pmatrix} \bar{v}_{23} \\ \bar{w}_{23} \\ \bar{\theta}_{23} \end{pmatrix} + \text{c.c.} + \epsilon^{\frac{6}{5}} A_{31} E \begin{pmatrix} \bar{v}_{31} \\ \bar{w}_{31} \\ \bar{\theta}_{31} \end{pmatrix} + \ldots,$$

where $E \equiv \exp\{\mathrm{i}k_c y\}$. The amplitude equation, obtained by considering terms in $\epsilon^{\frac{6}{5}}E$, is

$$\alpha_1 A_L |A_L|^2 - \alpha_2' \frac{\mathrm{d}^4 A_L}{\mathrm{d}X_L^4} + \alpha_3 X_L A_L + \alpha_6 Re_L \frac{\mathrm{d}A_L}{\mathrm{d}X_L} = 0, \tag{3.9}$$

where $\alpha_2' = \alpha_2/4k_c^2$, α_1, α_2 and α_3 are identical with those in the amplitude equation (3.3) for the transverse mode and α_6 is a negative constant whose value is given in

Appendix A. Again the coefficients are real, which leads us to look for solutions in which A_L is real. The boundary conditions are

$$A_L,\ \frac{dA_L}{dX_L}\to 0 \quad \text{as } X_L\to-\infty,$$

$$A_L \sim \left(-\frac{\alpha_3 X_L}{\alpha_1}\right)^{\frac{1}{2}},\quad \frac{dA_L}{dX_L}\sim\frac{1}{2}\left(\frac{-\alpha_3}{\alpha_1 X_L}\right)^{\frac{1}{2}} \quad \text{as } X_L\to\infty.$$

On applying the transformation

$$X_L = (\alpha_2'\alpha_3^{-1})^{\frac{1}{5}}\bar{X}_L,\quad A_L = \left[\frac{-2}{\alpha_1}\alpha_2^{\frac{1}{5}}\alpha_3^{\frac{4}{5}}\right]^{\frac{1}{2}}\bar{A}_L$$

and writing $Re_L = -\alpha_6^{-1}\alpha_2'^{\frac{2}{5}}\alpha_3^{\frac{3}{5}}\delta$, we obtain

$$\frac{d^4\bar{A}_L}{d\bar{X}_L}-\bar{X}_L\bar{A}_L+\delta\frac{d\bar{A}_L}{d\bar{X}_L}+2\bar{A}_L^3 = 0. \tag{3.10}$$

The numerical solution of (3.10) has been given in I for $\delta = 0$. This solution holds for $Re_L \ll 1$ or $Re \ll \epsilon^{\frac{3}{5}}$, and also for antisymmetric Couette flow ($U = z-\frac{1}{2}$), in which case $\alpha_6 = 0$. There is no difficulty in obtaining solutions of (3.10) subject to the boundary conditions

$$\left.\begin{aligned} &\bar{A}_L,\ \frac{d\bar{A}_L}{d\bar{X}_L}\to 0 \quad \text{as } \bar{X}_L\to-\infty,\\ &\bar{A}_L\sim(\tfrac{1}{2}\bar{X}_L)^{\frac{1}{2}},\quad \frac{d\bar{A}_L}{d\bar{X}_L}\sim\tfrac{1}{2}(2\bar{X}_L)^{-\frac{1}{2}} \quad \text{as } \bar{X}_L\to\infty,\end{aligned}\right\} \tag{3.11}$$

provided δ is not too large. As δ increases, the transition region is shifted to larger values of $\bar{X}_L$ and becomes steeper just as for the transverse mode, but we have been unable to find a solution of (3.10) and (3.11) for δ greater than a critical value δ_c, which we calculate to be 1.61. Three different numerical methods have been employed (expansion in Tchebycheff polynomials, shooting from either end or both ends of the range, and finite differences) and all are in close agreement: there is no solution for $\delta > 1.61$.

In order to find a way out of this impasse it is necessary to return to the formulation of the perturbation problem and re-examine the assumptions upon which it is based. The weakest part of the structure is the assumption that longitudinal rolls set in with a wavelength equal to that at the onset of unmodulated Bénard convection. Now if we consider a disturbance with wavenumber $k \neq k_c$ and such that $|k-k_c| = \delta k$ then, for δk small, the critical Rayleigh number needs to be increased by a term proportional to $(\delta k)^2$. This means that a disturbance of this kind cannot occur until $X \sim (\delta k)^2$. Clearly if we are primarily concerned with the mode that appears first, in the sense that it appears at the smallest possible value of X, then we need to choose the mode that minimizes $(\delta k)^2$. For $\delta < \delta_c$ we have obtained solutions with $\delta k = 0$, and that is therefore the preferred mode. For $\delta > \delta_c$ there is no solution for $\delta k = 0$ and we are led to consider solutions in which $\delta k \neq 0$.

Small departures of k from k_c may easily be encompassed within the present theory. If $|\delta k| \ll \epsilon^{\frac{2}{5}}$ we find that the amplitude equation (obtained by considering terms $O(\epsilon^{\frac{6}{5}})$) remains unchanged from that given earlier for $\delta k = 0$, but new terms appear in the equation if $\delta k \sim \epsilon^{\frac{2}{5}}$. Let us write

$$k = k_c + \epsilon^{\frac{2}{5}}k_1.$$

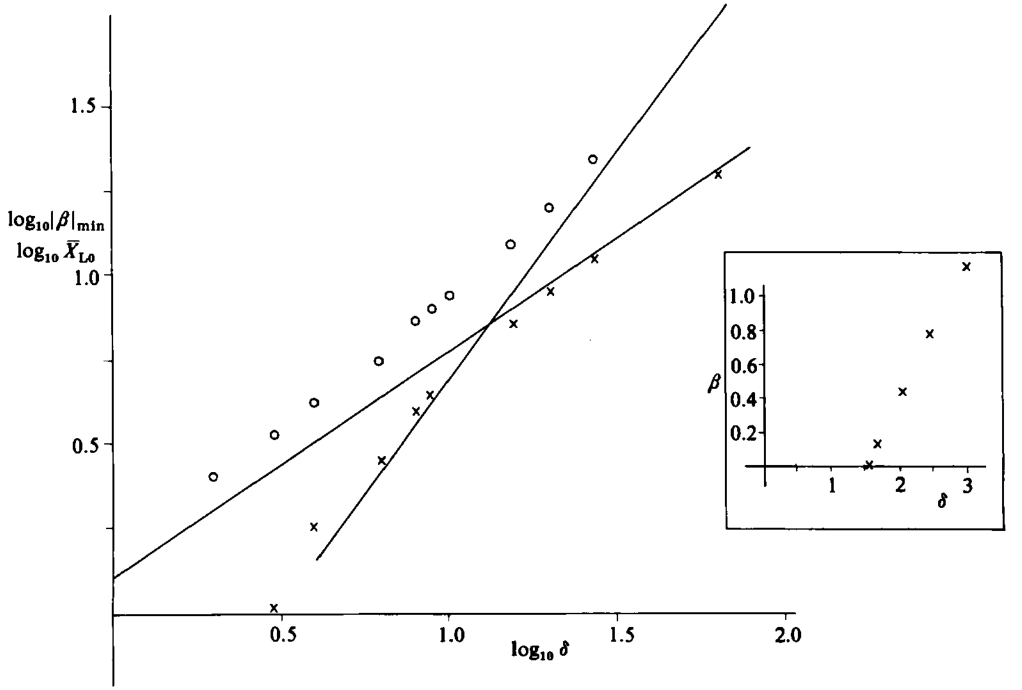

FIGURE 6. The minimum value of $|\beta|$ (points marked $\times$) for which (3.12) has solutions as a function of δ. The values of $\bar{X}_{L0}$ are denoted by $\odot$. The asymptotes shown are $|\beta|_{min} = 1.32\delta^{\frac{2}{3}}$ and $\bar{X}_{L0} = 0.24\delta^{\frac{4}{3}}$ respectively. The variation of $|\beta|_{min}$ with δ for δ in $[0, 3]$ is shown in the inset.

Then the expansion in powers of $\epsilon^{\frac{2}{5}}$ remains as above except that in the equations satisfied by $\bar{w}_1, \bar{\theta}_1$, etc.:

$$\nabla^2 = \left(\frac{\partial^2}{\partial z^2} - k_c^2\right) + \epsilon^{\frac{2}{5}}\left(\frac{\partial^2}{\partial X_L^2} - 2k_c k_1\right) + O(\epsilon^{\frac{4}{5}}).$$

As far as the amplitude equation is concerned, this means that we need to replace d^2/dX_L^2 with $d^2/dX_L^2 - 2k_c k_1$. The resulting amplitude equation is, after rescaling,

$$\frac{d^4\bar{A}_L}{d\bar{X}_L^4} - \bar{X}_L\bar{A}_L + \delta\frac{d\bar{A}_L}{d\bar{X}_L} - \beta\frac{d^2\bar{A}_L}{d\bar{X}_L^2} + 2\bar{A}_L^3 = 0, \tag{3.12}$$

where $$\beta = 4k_c k_1 (\alpha_2' \alpha_3^{-1})^{\frac{2}{5}}$$

and $$X_L = (\alpha_2'\alpha_3^{-1})^{\frac{1}{5}}[\bar{X}_L + \tfrac{1}{4}\beta^2].$$

The form of the scaling indicates that the solution is shifted to larger values of X_L by a term proportional to k_1^2, in agreement with our earlier discussion. In order to find the mode that appears at the lowest possible value of X_L, we need to find the minimum value of k_1^2 (and hence $|\beta|$) for which solutions exist to (3.12) and the appropriate boundary conditions (3.11).

Numerical integration reveals that $|\beta|_{min} = 0$ for $\delta < \delta_c = 1.61$ and increases monotonically with δ as shown in figure 6 for $\delta > \delta_c$. Sample solutions of (3.12), shown in figure 7, demonstrate that once again the onset of the disturbance occurs at larger values of X_L as δ increases and the transition regions becomes narrower. For $\delta \gg 1$ it appears that all five terms in (3.12) are important. A balance may be achieved by writing

$$\bar{X}_L = \delta^{\frac{4}{3}}\bar{X}_{L0} + \delta^{-\frac{1}{3}}\bar{X}_{L1}, \quad \bar{A}_L = \delta^{\frac{2}{3}}\bar{A}_{L0}, \quad |\beta|_{min} = |\beta_0|\delta^{\frac{2}{3}},$$

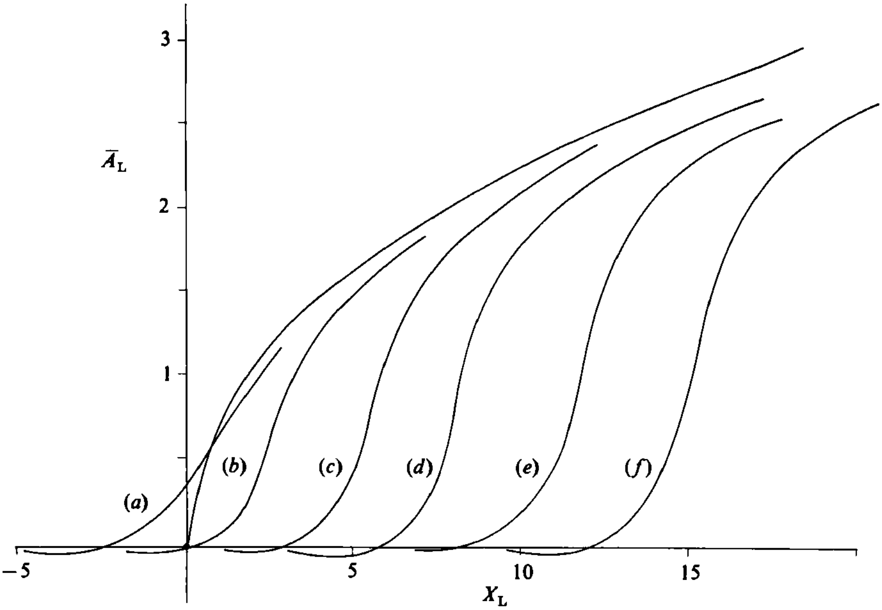

FIGURE 7. The amplitude $\bar{A}_L$ given by (3.12) for values of δ as follows: (a) 0; (b) 2; (c) 4; (d) 6; (e) 8; (f) 10.

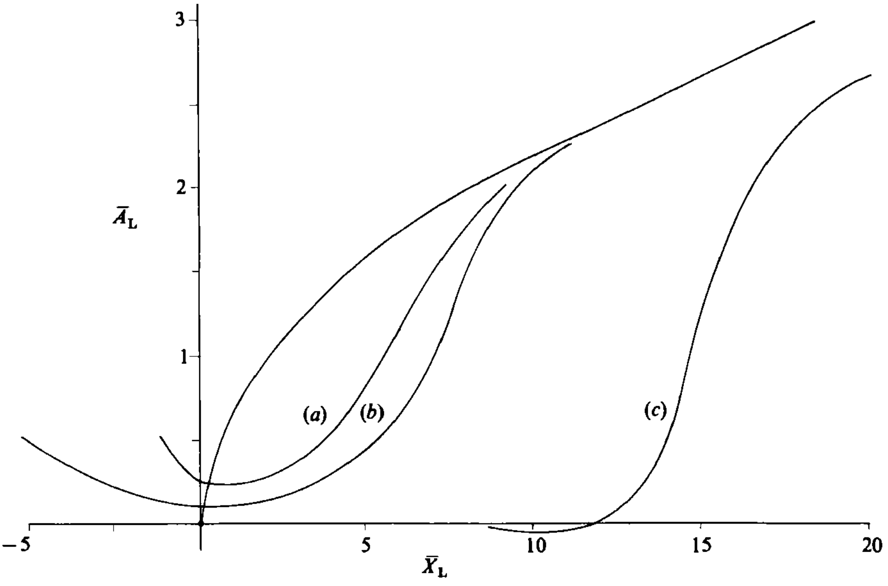

FIGURE 8. The amplitude $\bar{A}_L$ for $\delta = 10$ and various boundary conditions: (a) $\bar{A}_L = 0.5$, $\mathrm{d}\bar{A}_L/\mathrm{d}\bar{X}_L = -0.5$ at $\bar{X}_L = -1.0$; (b) $\bar{A}_L = 0.5$, $\mathrm{d}\bar{A}_L/\mathrm{d}\bar{X}_L = -0.4$ at $\bar{X}_L = -5.0$; (c) $\bar{A}_L \to 0$, $\mathrm{d}\bar{A}_L/\mathrm{d}\bar{X}_L \to 0$ as $\bar{X}_L \to -\infty$. In (a) and (b) we have used $\beta = 0$, and in (c) $\beta = 4.93$, the minimum value of β for which solutions of this kind exist.

in which case $\bar{A}_{L0}$ satisfies

$$\frac{\mathrm{d}^4\bar{A}_{L0}}{\mathrm{d}\bar{X}_{L1}^4} - \bar{X}_{L0}\,\bar{A}_{L0} + \frac{\mathrm{d}\bar{A}_{L0}}{\mathrm{d}\bar{X}_{L1}} - \beta_0\frac{\mathrm{d}^2\bar{A}_{L0}}{\mathrm{d}\bar{X}_{L1}^2} + 2\bar{A}_{L0}^3 = 0.$$

The boundary conditions on $\bar{A}_{L0}$ are

$$\bar{A}_{L0},\quad \frac{d\bar{A}_{L0}}{d\bar{X}_{L1}} \to 0 \quad \text{as } \bar{X}_{L1} \to -\infty,$$

$$\bar{A}_{L0} \to (\tfrac{1}{2}\bar{X}_{L0})^{\frac{1}{2}},\quad \frac{dA_{L0}}{d\bar{X}_{L1}} \to 0 \quad \text{as } \bar{X}_{L1} \to +\infty.$$

Numerical integration of (3.12) shows that $\bar{X}_{L0} = 0.24$ and $\beta_0 = 1.32$.

These results indicate that the horizontal extent of the transition region diminishes like $\delta^{-\frac{1}{3}}$ as $\delta \to \infty$, which is equivalent to $\epsilon^{\frac{1}{5}}\, Re^{-\frac{1}{3}}$ on the scale of $\bar{X}_L$ or $Re^{-\frac{1}{3}}$ on the scale of x. As for transverse rolls, when Re is as large as unity the transition takes place within a few wavelengths, and the method of multiple scales is no longer applicable. Also the wavenumber of the disturbance increases like $\epsilon^{\frac{2}{5}}\delta^{\frac{2}{3}}$, or $Re^{\frac{2}{3}}$, which means that the assumption that the wavenumber remains close to critical also breaks down when $Re \sim 1$. Furthermore, transition occurs at a point $X \sim Re^{\frac{4}{3}}$ for $\delta \gg 1$.

4. Discussion

We have shown that a weak shear flow has three main effects on the onset of convection in a non-uniformly heated layer. First, any significant growth of convection is delayed, in that it occurs further downstream. Secondly, the onset is more sudden, in that the rise in amplitude from a very small value to a value close to the asymptotic form far downstream takes place over a narrower interval. Thirdly, in the case of longitudinal rolls the wavenumber is perturbed away from the critical value that holds in the absence of a shear flow. For shear-flow Reynolds numbers of the magnitude considered here the onset of the longitudinal mode occurs at smaller values of X than the transverse mode.

Our analysis can take into account shear-flow Reynolds numbers $Re \lesssim 1$. The solutions for $Re \sim 1$ and $Re \gtrsim 1$ remain to be found, and may well be quite different in structure from those discussed here. Our results suggest that for larger values of Re the onset of convection occurs at a considerable distance downstream from the point of neutral stability based on linear theory, and that the onset takes place over a very short distance. It seems likely that a much fuller set of elliptic partial differential equations needs to be solved in this region.

An important feature of our results is that we have been successful in obtaining solutions that satisfy the boundary conditions that hold when the instability is triggered by an infinitesimal disturbance or by an initial small but finite-amplitude disturbance. Our formulation also allows us to treat a problem analogous to that posed by Hall & Smith (1984) and Hall (1982), concerning the effect of a disturbance of small but finite amplitude maintained at some point $X = X^*$, say, for all time. If we follow those authors in assuming that the effects of diffusion in the X-direction may be neglected, then a scaling that leads to an equation of the form (1.3) is $X \sim (\epsilon\, Re)^{\frac{1}{2}}$, $A \sim (\epsilon\, Re)^{\frac{1}{4}}$. An examination of the leading terms neglected in this approximation reveals that it remains valid only for $Re \gg \epsilon^{\frac{1}{3}}$ in the case of a transverse disturbance and for $Re \gg \epsilon^{\frac{3}{5}}$ for a longitudinal disturbance. In other words, diffusion in the X-direction may be neglected only if the shear flow is sufficiently strong or the rate of variation in the thermal forcing sufficiently weak. Our analysis allows us to extend this restrictive parameter range to include $Re \lesssim \epsilon^{\frac{1}{3}}$ and $Re \lesssim \epsilon^{\frac{3}{5}}$ for transverse and longitudinal disturbances respectively, in which case diffusion in the X-direction is as important as the other terms in (1.3). Sample results, with A, dA/dX

prescribed at some station $X = X^*$, are shown in figure 8. A noteworthy feature is that, although the point of neutral stability is shifted slightly to larger values of X than predicted on the basis of linear theory, the shift for $O(1)$ values of the scaled amplitude of the disturbance is very much less than for an infinitesimal value. These results suggest that the onset of convection can be delayed a considerable distance downstream by keeping the amplitude of the forced disturbance small. In this respect, observations of the onset of convection due to a finite forced disturbance are a poor guide to the true stability properties of the flow.

Appendix A. Numerical values of the coefficients in the amplitude equations

For the transverse mode the amplitude satisfies (3.3) in which

$$\alpha_1 = -4.5118, \quad \alpha_2 = 1.8163, \quad \alpha_3 = 12.143\frac{F'(0)}{R_c},$$

$$\alpha_4 = \begin{cases} -(3.9810Pr^2+0.9017Pr+0.4586)\,10^{-3} & \text{(Couette flow)}, \\ -\dfrac{(0.5024Pr^4+0.4246Pr^3+2.635Pr^2+0.0243Pr+0.0253)10}{(0.6178Pr+0.3162)^2} & \text{(Poiseuille flow)}, \end{cases}$$

$$\alpha_5 = \begin{cases} 0 & (\text{Couette flow, } U = z-\tfrac{1}{2}), \\ -(0.1581+0.3089Pr) & (\text{Couette flow, } U = z), \\ -\dfrac{(0.1862+0.5425Pr)-2k_c^2 10^{-2}(0.0046Pr^2+0.2970Pr-0.0005)}{0.6178Pr+0.3162} & \text{(Poiseuille flow)}. \end{cases}$$

Also, the leading approximation to the wave speed c_0 is zero for antisymmetric Couette flow and $\frac{1}{2}$ for Couette flow with $U = z$; in both cases c_0 is equal to the speed of the shear flow at the midpoint of the fluid layer. For Poiseuille flow we find that

$$c_0 = \frac{0.1862+0.5425Pr}{0.3162+0.6178Pr}.$$

For the longitudinal mode the coefficient α_6 in (3.9) takes the values

$$\alpha_6 = \begin{cases} 0 & (\text{Couette flow, } U = z-\tfrac{1}{2}), \\ -(0.1581+0.3089Pr) & (\text{Couette flow, } U = z), \\ -(0.1862+0.5425Pr) & (\text{Poiseuille flow}). \end{cases}$$

Appendix B. Solution of (3.6) for $\delta \gg 1$

The solution is divided into three regions: (a) $\bar{X}_T \ll \bar{X}_{T0}\delta^2$, (b) $|\bar{X}_T - \bar{X}_{T0}\delta^2| \ll 1$ and (c) $\bar{X}_T \gg \bar{X}_{T0}\delta^2$. The transition region (b) has already been discussed in §3. In region (a) the amplitude $\bar{A}_T$ is small, and we may neglect the nonlinear term. The truncated version of (3.6) has the solution

$$\bar{A}_T = \text{const} \times \exp(\tfrac{1}{2}\delta\bar{X}_T)\,\text{Ai}\,(\tfrac{1}{4}\delta^2 - \bar{X}_T) \tag{B 1}$$

where Ai is an Airy function. For δ fixed, $\bar{A}_T \to 0$ as $\bar{X}_T \to -\infty$. In terms of the transition-region variable $\bar{X}_T$ we have

$$\bar{A}_T = \text{const} \times \exp(\tfrac{1}{2}\delta^3\bar{X}_{T0} + \tfrac{1}{2}\bar{X}_{T1})\,\text{Ai}\,(\delta(\tfrac{1}{4} - \bar{X}_{T0}) - \delta^{-1}\bar{X}_{T1}),$$

where $\bar{X}_{T1} < 0$. The limit $\delta \to \infty$ clearly depends upon the sign of $\frac{1}{4} - \bar{X}_{T0}$. If $\bar{X}_{T0} < \frac{1}{4}$ we find that

$$\bar{A}_T \sim \exp\{[\tfrac{1}{2} + (\tfrac{1}{4} - \bar{X}_{T0})^{\frac{1}{2}}]\,\bar{X}_{T1}\} \quad \text{as } \delta \to \infty, \tag{B 2}$$

while if $\bar{X}_{T0} > \frac{1}{4}$ we find that

$$\bar{A}_T \sim \exp(\tfrac{1}{2}\bar{X}_{T1}) \cos[(\bar{X}_{T0} - \tfrac{1}{4})^{\frac{1}{2}}\,\bar{X}_{T1} + \beta(\delta)] \quad \text{as } \delta \to \infty. \tag{B 3}$$

Equation (B 2) or (B 3) provides the boundary condition for (3.7) as $\bar{X}_{T1} \to -\infty$. Similar expressions may be obtained in the same way for $\bar{X}_{T1} \to \infty$. It is clear now that (3.8) is not the required solution, for it predicts that $\bar{A}_T \sim \exp \frac{1}{3}\bar{X}_{T1}$ as $\bar{X}_{T1} \to -\infty$, while, with $\bar{X}_{T0} = \frac{2}{9}$, (B 2) demands $\bar{A}_T \sim \exp \frac{2}{3}\bar{X}_{T1}$ as $\bar{X}_{T1} \to -\infty$. We have been unable to find a solution of (3.7) with boundary conditions in the form (B 2) appropriate to $\bar{X}_{T0} < \frac{1}{4}$.

The difficulty with $\bar{X}_{T0} > \frac{1}{4}$ arises from the asymptotic behaviour of (B 1) and the corresponding solution with Ai replaced by Bi. These expressions differ only by an $O(1)$ term in the phase $\beta(\delta)$ in (B 3). Since $\beta(\delta) \sim \delta^3$ we cannot detect this difference to leading order in δ, and we are unable therefore to specify boundary conditions on $\bar{A}_{T0}$ in the transition region that distinguish between (B 1) and the corresponding solution with Ai replaced by Bi.

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